Distribution of zero subsequences for Bernstein space and criteria of completeness for exponential system on a segment

Bulat N. Khabibullin*

Abstract

For $\sigma \in (0,+\infty)$, denote by B_σ^∞ the Bernstein space (of type σ) of all entire functions of exponential type $\leq \sigma$ bounded on real axis $\mathbb{R}$. Let $I_d \subset \mathbb{R}$ be a segment of length $d>0$. We announce complete description of non-uniqueness sequences of points for B_σ^∞ and criteria of completeness of exponential system in $C(I_d)$ or $L^p(I_d)$ accurate within one or two exponential functions.

Keywords: entire function, Bernstein space, zero subsequence, uniqueness sequence, completeness, exponential system, Poisson integral, Hilbert transform

1 Definitions, and statements of problems

Denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{C}$ the sets of natural, real, integer, and complex numbers respectively. Besides, $\mathbb{R}_* := \mathbb{R}\setminus\{0\}$, $\mathbb{C}_* := \mathbb{C}\setminus\{0\}$, $\mathbb{C}_\pm := \mathbb{C}\setminus\mathbb{R}$.

For a segment $I_d \subset \mathbb{R}$ of length d, we denote by $C(I_d)$ and $L^p(I_d)$ the space of continuous functions f on I_d with sup-norm $\|f\|_\infty := \sup\bigl\{|f(x)| \colon x \in I_d\bigr\}$ and the space of functions f with finite norm

$$\|f\|_p := \left(\int_{I_d} |f(x)|^p \, dx\right)^{1/p}, \quad p \geq 1.$$

For $\sigma \in (0,+\infty)$, denote by B_σ^∞ *the Bernstein space* (of type σ) of all entire functions of exponential type $\leq \sigma$ bounded on $\mathbb{R}$, i. e.

$$\log|f(z)| \leq \sigma|\operatorname{Im} z| + c_f, \quad z \in \mathbb{C}, \tag{1.1}$$

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where c_f is a constant (see [1]).

Let

$$\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C} \tag{1.2}$$

be a point sequence on $\mathbb{C}$ without limit points in $\mathbb{C}$. The sequence Λ is a *zero subsequence* (non-uniqueness sequence) for B_σ^∞ iff there exists a nonzero function $f_\Lambda \in B_\sigma^\infty$ such that f_Λ vanish on Λ, i. e. $f_\Lambda(\lambda_k) = 0$ for $k \in \mathbb{N}$. Each sequence Λ defines a *counting measure*

$$n_\Lambda(S) := \sum_{\lambda_k \in S} 1, \quad S \subset \mathbb{C}, \tag{1.3}$$

and an *exponential system*

$$\operatorname{Exp}^\Lambda := \{x \mapsto x^{m-1} e^{\lambda x} \colon x \in I_d,\ \lambda \in \Lambda,\ 1 \le m \le n_\Lambda\bigl(\{\lambda\}\bigr),\ m \in \mathbb{N}\}. \tag{1.4}$$

The exponential system $\operatorname{Exp}^{i\Lambda}$, $i\Lambda := \{i\lambda_k\}_{k\in\mathbb{N}}$, is *complete* in $C(I_d)$ or $L^p(I_d)$ iff the closure of its linear span coincides with $C(I_d)$ or $L^p(I_d)$.

We solve the following problems.

- *Complete description of zero subsequences for B_σ^∞.*
- *Criteria of completeness of exponential system* $\operatorname{Exp}^{i\Lambda}$ *in $C(I_d)$ or $L^p(I_d)$ accurate within one or two exponential functions.*

It was many times noticed that obtaining of criteria of completeness of exponential systems $\operatorname{Exp}^{i\Lambda}$ in $C(I_d)$ or $L^p(I_d)$ purely in terms Λ and d is exclusively difficult problem [2], [3], [4, p. 167], [5, 3.3], [6, 0.1].

Poisson integral. For $z \in \mathbb{C}_\pm$ *the Poisson integral* $\mathrm{P}_{\mathbb{C}_\pm}\varphi$ of function $\varphi \in L^1(\mathbb{R})$ is defined as

$$\begin{aligned}(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(z) :&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\operatorname{Im} z|}{(t-\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}\,\varphi(t)\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \left|\operatorname{Im}\frac{1}{t-z}\right| \varphi(t)\,dt,\end{aligned} \tag{1.5}$$

But for $x \in \mathbb{R}$ we set

$$(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(x) := \varphi(x), \quad x \in \mathbb{R}_*. \tag{1.6}$$

The Poisson integral from (1.5) is harmonic function on $\mathbb{C}_\pm$.

If $\varphi \in C(\mathbb{R}_*)$, then the Poisson transform $\mathrm{P}_{\mathbb{C}_\pm}\varphi$ is also continuous function on $\mathbb{C}_*$. More in detail look in [7].

Hilbert transform (see [7]**).** For a function

$$\varphi \in L^1(\mathbb{R}_*), \quad \varphi\colon \mathbb{R}_* \to \mathbb{R},$$

the *direct Hilbert transform* H is defined as a rule by integral

$$(\mathrm{H}\,\varphi)(x) := \frac{1}{\pi} \fint_{\mathbb{R}_*} \frac{\varphi(t)}{x-t}\, dt, \quad x \in \mathbb{R}_*,$$

where the strikethrough of integral

$$\fint_{\mathbb{R}_*} := \lim_{0<\varepsilon\to 0} \int_{\mathbb{R}_* \setminus (x-\varepsilon, x+\varepsilon)}, \quad x \neq 0,$$

means the principal integral value in the sense of Cauchy. Such function $\mathrm{H}\,\varphi$ is good defined almost everywhere on $\mathbb{R}$. The *inverse Hilbert transform* differs only by sign

$$(\mathrm{H}^{-1}\,\varphi)(x) := \frac{1}{\pi} \fint_{\mathbb{R}_*} \frac{\varphi(t)}{t-x}\, dt = -(\mathrm{H}\,\varphi)(x), \quad x \in \mathbb{R}_*.$$

2 Main Theorem

Classes of test functions $R\mathcal{P}_0^m$, $2 \le m \le \infty$. This class will consist of all *positive continuous functions*

$$\varphi\colon \mathbb{R}_* \to [0,+\infty), \quad Z_\varphi := \{x \in \mathbb{R}_*\colon \varphi(x) = 0\},$$

from the class $C^m(\mathbb{R}_* \setminus Z_\varphi)$ of m times continuously differentiable functions on $\mathbb{R}_* \setminus Z_\varphi$ with

- *a finiteness condition*

$$\varphi(x) \equiv 0, \quad |x| \ge R_\varphi > 0, \tag{2.1}$$

  where $R_\varphi > 0$ is a constant;

- *a semi-normalization condition*

$$\limsup_{0\neq x\to 0} \frac{\varphi(x)}{\log\bigl(1/|x|\bigr)} \le 1; \tag{2.2}$$

- *a conjugate condition of positivity*

$$(-\mathrm{H}\,\varphi)'(x) := \frac{1}{\pi} \fint_{\mathbb{R}_*} \frac{\varphi(t)-\varphi(x)}{(t-x)^2}\,dt \geq 0, \quad x \in \mathbb{R}_* \setminus Z_\varphi, \tag{2.3}$$

  i. e. condition of decrease for the Hilbert transform $\mathrm{H}\,\varphi$ separately on every connected component (open interval) of the subset $\mathbb{R}_* \setminus Z_\varphi \subset \mathbb{R}$.

The Hilbert transform and the differentiation operator commute, i. e.

$$\frac{d}{dx}(-\mathrm{H}\,\varphi) \equiv -\mathrm{H}\,\frac{d}{dx}\varphi,$$

for $\varphi \in L^p(\mathbb{R}_*)$, $p \geq 1$, and for Schwartz distributions φ (see [8]).

Besides, the left-hand member of (2.3) can be rewritten as

$$(-\mathrm{H}\,\varphi)'(x) = \frac{1}{\pi} \fint_0^{+\infty} \frac{\varphi(x+t)+\varphi(x-t)-2\varphi(x)}{t^2}\,dt.$$

Main Theorem (on zero subsequence for Bernstein space)**.** *Let* $\Lambda \not\ni 0$ *be a point sequence on* $\mathbb{C}$ *from* (1.2)*, and* $\sigma \in (0, +\infty)$*. The following three assertions are equivalent:*

1) Λ *is zero subsequence for Bernstein space* B^∞_σ*;*
2) *for the some (for any)* $m \in \bigl(\mathbb{N} \setminus \{1\}\bigr) \cup \infty$ *the condition*

$$\sup_{\varphi \in R\mathcal{P}_0^m} \left(\sum_{k \in \mathbb{N}} (\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) - \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} \varphi(x)\,dx \right) < +\infty \tag{2.4}$$

   is fulfilled.
3) *the condition of* (2.4) *is fulfilled after replacement of the class* $R\mathcal{P}_0^m$ *with* $R\mathcal{P}_0^m \cap C^\infty(\mathbb{R}_*)$*, where* $C^\infty(\mathbb{R}_*)$ *is the class of infinitely differentiable functions on* $\mathbb{R}_*$*.*

Remark 1. If $\Lambda \subset \mathbb{R}_*$ is real sequence, then (2.4) looks absolutely simply:

$$\sup_{\varphi \in \mathcal{RP}_0^m} \left(\sum_{\lambda \in \Lambda} \varphi(\lambda) - \frac{\sigma}{\pi} \int_{-\infty}^{+\infty} \varphi(x)\,dx \right) < +\infty. \tag{2.5}$$

Remark 2. Any shifts of any finite number of points from Λ do not change his property as zero subsequence for B^∞_σ. Therefore, without loss of generality, we can consider only cases $0 \notin \Lambda$.

Remark 3. Conditions (2.4)and (2.5) are similar to definition of order relation on sets of real-valued Radon measures or on distributions. Indeed, if ν and μ are two such measures or two distributions on $\mathbb{C}$, then $\nu \leq \mu$ iff

$$\sup_{\varphi\in (C_0^\infty(\mathbb{C}))^+} \bigl(\nu(\varphi) - \mu(\varphi)\bigr) \leq 0, \tag{2.6}$$

where $(C_0^\infty(\mathbb{C}))^+$ is the class of infinitely differentiable finite positive functions on $\mathbb{C}$.

The measure μ_σ with density

$$d\mu_\sigma(x) := \frac{\sigma}{\pi}\, dx, \quad x \in \mathbb{R},$$

on $\mathbb{R}$ is the Riesz measure of subharmonic function

$$M_\sigma(z) := \sigma|\operatorname{Im} z|, \quad z \in \mathbb{C}, \quad \mu_\sigma := \frac{1}{2\pi}\, \Delta M_\sigma \geq 0,$$

where Δ is the Laplace operator in the sense of the Schwartz distribution theory. The function M_σ is located in the right-hand member of the inequality (1.1) and defines the Bernstein space B_σ^∞. In these notations the conditions (2.4) and (2.5) it is possible to write in the form (see (1.3))

$$\sup_{\varphi\in R\mathcal{P}_0^m} \bigl(n_\Lambda(\mathrm{P}_{\mathbb{C}_\pm}\, \varphi) - \mu_\sigma(\varphi)\bigr) < +\infty.$$

This record is useful for comparison to (2.6).

3 On the completeness of exponential systems

A system of vectors of vector topological space is complete in this space if the closure of its linear span coincides with this space. Otherwise the system is incomplete.

For spaces on a segment or an interval on $\mathbb{R}$ traditionally consider exponential systems with sequence of exponents

$$i\Lambda := \{i\lambda_k\}_{k\in\mathbb{N}}, \quad \operatorname{Exp}^{i\Lambda} = \bigl\{e^{i\lambda x} \colon \lambda \in \Lambda,\ x \in \mathbb{R}\bigr\}.$$

Denote by $I_d \subset \mathbb{R}$ an arbitrary segment $[a, b]$ of length $d = b - a$.

By virtue of known interrelation between uniqueness and completeness the Main Theorem implies

Theorem 1 (on completeness of exponential systems)**.** *If a point sequence* $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \not\ni 0$ *satisfies to the condition*

$$\sup_{\varphi\in R\mathcal{P}_0^m}\left(\sum_{k\in\mathbb{N}}(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) - \frac{d}{2\pi}\int_{-\infty}^{+\infty}\varphi(x)\,dx\right) = +\infty, \tag{3.1}$$

then the system $\mathrm{Exp}^{i\Lambda}$ *is complete in* $C(I_d)$ *and* $L^p(I_d)$*.*

Inversely, if

$$\sup_{\varphi\in R\mathcal{P}_0^m}\left(\sum_{k\in\mathbb{N}}(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) - \frac{d}{2\pi}\int_{-\infty}^{+\infty}\varphi(x)\,dx\right) < +\infty, \tag{3.2}$$

then, for any pair different points $\{\lambda', \lambda''\} \subset \Lambda$*, the system* $\mathrm{Exp}^{i\Lambda\setminus\{i\lambda'\}}$ *is incomplete in* $C(I_d)$ *and* $L^p(I_d)$ *for* $p \geq 2$*, and the system* $\mathrm{Exp}^{i\Lambda\setminus\{i\lambda', i\lambda''\}}$ *is incomplete in* $L^p(I_d)$ *for* $1 \leq p < 2$*.*

Remark 4. From Theorem 1 we can obtain a whole number of basic old results on completeness of exponential systems in spaces $C(I_d)$ and in $L^p(I_d)$ (for example, the Berling–Malliavin Theorem on radius of completeness), and also new results.

Example 1. Let's prove very briefly and quickly one old result.

Theorem 2 ([9], [10, Theorem 41])**.** *If a point sequence* $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *satisfies to two conditions*

$$0 < \alpha \leq |\arg\lambda_k| \leq \pi - \alpha,\ k \in \mathbb{N}, \qquad \sum_{k\in\mathbb{N}}\left|\mathrm{Im}\,\frac{1}{\lambda_k}\right| < +\infty\,, \tag{3.3}$$

then the system $\mathrm{Exp}^{i\Lambda}$ *is incomplete in any* $C(I_d)$ *and* $L^p(I_d)$*.*

Proof. Without loss generality we can consider, that for arbitrary ε the condition

$$\sum_{k\in\mathbb{N}}\left|\mathrm{Im}\,\frac{1}{\lambda_k}\right| < \varepsilon \tag{3.4}$$

is fulfilled instead of second condition from (3.3). For this purpose it is enough to shift finite number of points. It does not change the property of (in-)completeness (see Remark 2). First condition from (3.3) implies the

estimate $|\operatorname{Im}\lambda_k| \geq |\lambda_k| \sin\alpha$. This estimate give the following estimate for the Poisson kernel at every point λ_k:

$$\frac{1}{\pi} \frac{|\operatorname{Im}\lambda_k|}{(t-\operatorname{Re}\lambda_k)^2 + (\operatorname{Im}\lambda_k)^2} \leq \frac{1}{\pi\sin^2\alpha} \frac{|\operatorname{Im}\lambda_k|}{|\lambda_k|^2} = \frac{1}{\pi\sin^2\alpha} \left|\operatorname{Im}\frac{1}{\lambda_k}\right|.$$

Hence in view of (3.4) for arbitrary $\varphi \in R\mathcal{P}_0^m$ we have

$$\sum_{k\in\mathbb{N}} (\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) := \sum_{k\in\mathbb{N}} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|\operatorname{Im}\lambda_k|}{(t-\operatorname{Re}\lambda_k)^2 + (\operatorname{Im}\lambda_k)^2}\,\varphi(t)\,dt$$
$$\leq \sum_{k\in\mathbb{N}} \frac{1}{\pi\sin^2\alpha}\left|\operatorname{Im}\frac{1}{\lambda_k}\right| \cdot \int_{-\infty}^{+\infty}\varphi(t)\,dt \leq \frac{\varepsilon}{\pi\sin^2\alpha} \cdot \int_{-\infty}^{+\infty}\varphi(x)\,dx.$$

If we choose ε sufficiently small, then

$$\frac{\varepsilon}{\pi\sin^2\alpha} \leq \frac{d'}{2\pi} < \frac{d}{2\pi},$$

and the condition (3.2) of Theorem 1 is fulfilled with d' instead of d. By Theorem 1 the exponential system $\operatorname{Exp}^{i\Lambda}$ is incomplete in spaces $C(I_{d'})$ and $L^p(I_{d'})$ without one or two functions, where $d' < d$. Thereby, the system $\operatorname{Exp}^{i\Lambda}$ is incomplete in spaces $C(I_d)$ and in $L^p(I_d)$ for any $d > 0$. •

Let's remind the Beurling–Malliavin Theorem in the Redheffer's interpretation (see [10, Theorem 77], [4]–[7]).

Beurling–Malliavin Theorem (on the radius of completeness)**.** *Let* $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$. *If there exists a number* $c > 0$ *and a sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of distinct integers such that the series*

$$\sum_{k\in\mathbb{N}} \left|\frac{1}{\lambda_k} - \frac{c}{2\pi n_k}\right| \tag{3.5}$$

converges, then the system $\operatorname{Exp}^{i\Lambda}$ *is incomplete in* $C(I_d)$ *and* $L^p(I_d)$ *for any* $d > c$. *Inversely, if the series* (3.5) *diverges, then the system* $\operatorname{Exp}^{i\Lambda}$ *is complete in* $C(I_d)$ *and* $L^p(I_d)$ *for any* $d < c$.

On the basis of Theorem 1 on completeness of exponential systems it is possible to give a new proof of the Beurling–Malliavin Theorem on the radius of completeness. But it is impossible to name this new proof neither short, nor simple.

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Remark 1. If $\Lambda \subset \mathbb{R}_*$ is real sequence, then (2.4) looks absolutely simply:

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Remark 3. Conditions (2.4)and (2.5) are similar to definition of order relation on sets of real-valued Radon measures or on distributions. Indeed, if ν and μ are two such measures or two distributions on $\mathbb{C}$, then $\nu \leq \mu$ iff

$$\sup_{\varphi \in (C_0^\infty(\mathbb{C}))^+} \bigl(\nu(\varphi) - \mu(\varphi)\bigr) \leq 0, \tag{2.6}$$

where $(C_0^\infty(\mathbb{C}))^+$ is the class of infinitely differentiable finite positive functions on $\mathbb{C}$.

The measure μ_σ with density

$$d\mu_\sigma(x) := \frac{\sigma}{\pi}\, dx, \quad x \in \mathbb{R},$$

on $\mathbb{R}$ is the Riesz measure of subharmonic function

$$M_\sigma(z) := \sigma|\operatorname{Im} z|, \quad z \in \mathbb{C}, \quad \mu_\sigma := \frac{1}{2\pi}\, \Delta M_\sigma \geq 0,$$

where Δ is the Laplace operator in the sense of the Schwartz distribution theory. The function M_σ is located in the right-hand member of the inequality (1.1) and defines the Bernstein space B_σ^∞. In these notations the conditions (2.4) and (2.5) it is possible to write in the form (see (1.3))

$$\sup_{\varphi \in R\mathcal{P}_0^m} \bigl(n_\Lambda(\mathrm{P}_{\mathbb{C}_\pm}\, \varphi) - \mu_\sigma(\varphi)\bigr) < +\infty.$$

This record is useful for comparison to (2.6).

3 On the completeness of exponential systems

A system of vectors of vector topological space is complete in this space if the closure of its linear span coincides with this space. Otherwise the system is incomplete.

For spaces on a segment or an interval on $\mathbb{R}$ traditionally consider exponential systems with sequence of exponents

$$i\Lambda := \{i\lambda_k\}_{k\in\mathbb{N}}, \quad \operatorname{Exp}^{i\Lambda} = \bigl\{e^{i\lambda x} \colon \lambda \in \Lambda,\ x \in \mathbb{R}\bigr\}.$$

Denote by $I_d \subset \mathbb{R}$ an arbitrary segment $[a, b]$ of length $d = b - a$.

By virtue of known interrelation between uniqueness and completeness the Main Theorem implies

Theorem 1 (on completeness of exponential systems)**.** *If a point sequence* $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \not\ni 0$ *satisfies to the condition*

$$\sup_{\varphi\in R\mathcal{P}_0^m}\left(\sum_{k\in\mathbb{N}}(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) - \frac{d}{2\pi}\int_{-\infty}^{+\infty}\varphi(x)\,dx\right) = +\infty, \tag{3.1}$$

then the system $\mathrm{Exp}^{i\Lambda}$ *is complete in* $C(I_d)$ *and* $L^p(I_d)$*.*

Inversely, if

$$\sup_{\varphi\in R\mathcal{P}_0^m}\left(\sum_{k\in\mathbb{N}}(\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) - \frac{d}{2\pi}\int_{-\infty}^{+\infty}\varphi(x)\,dx\right) < +\infty, \tag{3.2}$$

then, for any pair different points $\{\lambda', \lambda''\} \subset \Lambda$*, the system* $\mathrm{Exp}^{i\Lambda\setminus\{i\lambda'\}}$ *is incomplete in* $C(I_d)$ *and* $L^p(I_d)$ *for* $p \geq 2$*, and the system* $\mathrm{Exp}^{i\Lambda\setminus\{i\lambda', i\lambda''\}}$ *is incomplete in* $L^p(I_d)$ *for* $1 \leq p < 2$*.*

Remark 4. From Theorem 1 we can obtain a whole number of basic old results on completeness of exponential systems in spaces $C(I_d)$ and in $L^p(I_d)$ (for example, the Berling–Malliavin Theorem on radius of completeness), and also new results.

Example 1. Let's prove very briefly and quickly one old result.

Theorem 2 ([9], [10, Theorem 41])**.** *If a point sequence* $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *satisfies to two conditions*

$$0 < \alpha \leq |\arg\lambda_k| \leq \pi - \alpha,\ k \in \mathbb{N}, \qquad \sum_{k\in\mathbb{N}}\left|\mathrm{Im}\,\frac{1}{\lambda_k}\right| < +\infty\,, \tag{3.3}$$

then the system $\mathrm{Exp}^{i\Lambda}$ *is incomplete in any* $C(I_d)$ *and* $L^p(I_d)$*.*

Proof. Without loss generality we can consider, that for arbitrary ε the condition

$$\sum_{k\in\mathbb{N}}\left|\mathrm{Im}\,\frac{1}{\lambda_k}\right| < \varepsilon \tag{3.4}$$

is fulfilled instead of second condition from (3.3). For this purpose it is enough to shift finite number of points. It does not change the property of (in-)completeness (see Remark 2). First condition from (3.3) implies the

estimate $|\operatorname{Im}\lambda_k| \geq |\lambda_k| \sin\alpha$. This estimate give the following estimate for the Poisson kernel at every point λ_k:

$$\frac{1}{\pi}\,\frac{|\operatorname{Im}\lambda_k|}{(t-\operatorname{Re}\lambda_k)^2+(\operatorname{Im}\lambda_k)^2} \leq \frac{1}{\pi\sin^2\alpha}\,\frac{|\operatorname{Im}\lambda_k|}{|\lambda_k|^2} = \frac{1}{\pi\sin^2\alpha}\left|\operatorname{Im}\frac{1}{\lambda_k}\right|.$$

Hence in view of (3.4) for arbitrary $\varphi \in R\mathcal{P}_0^m$ we have

$$\sum_{k\in\mathbb{N}} (\mathrm{P}_{\mathbb{C}_\pm}\varphi)(\lambda_k) := \sum_{k\in\mathbb{N}} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{|\operatorname{Im}\lambda_k|}{(t-\operatorname{Re}\lambda_k)^2+(\operatorname{Im}\lambda_k)^2}\,\varphi(t)\,dt$$
$$\leq \sum_{k\in\mathbb{N}} \frac{1}{\pi\sin^2\alpha}\left|\operatorname{Im}\frac{1}{\lambda_k}\right| \cdot \int_{-\infty}^{+\infty}\varphi(t)\,dt \leq \frac{\varepsilon}{\pi\sin^2\alpha}\cdot\int_{-\infty}^{+\infty}\varphi(x)\,dx.$$

If we choose ε sufficiently small, then

$$\frac{\varepsilon}{\pi\sin^2\alpha} \leq \frac{d'}{2\pi} < \frac{d}{2\pi}\,,$$

and the condition (3.2) of Theorem 1 is fulfilled with d' instead of d. By Theorem 1 the exponential system $\operatorname{Exp}^{i\Lambda}$ is incomplete in spaces $C(I_{d'})$ and $L^p(I_{d'})$ without one or two functions, where $d' < d$. Thereby, the system $\operatorname{Exp}^{i\Lambda}$ is incomplete in spaces $C(I_d)$ and in $L^p(I_d)$ for any $d > 0$. •

Let's remind the Beurling–Malliavin Theorem in the Redheffer's interpretation (see [10, Theorem 77], [4]–[7]).

Beurling–Malliavin Theorem (on the radius of completeness)**.** *Let* $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$. *If there exists a number* $c > 0$ *and a sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of distinct integers such that the series*

$$\sum_{k\in\mathbb{N}} \left|\frac{1}{\lambda_k} - \frac{c}{2\pi n_k}\right| \tag{3.5}$$

converges, then the system $\operatorname{Exp}^{i\Lambda}$ *is incomplete in* $C(I_d)$ *and* $L^p(I_d)$ *for any* $d > c$. *Inversely, if the series* (3.5) *diverges, then the system* $\operatorname{Exp}^{i\Lambda}$ *is complete in* $C(I_d)$ *and* $L^p(I_d)$ *for any* $d < c$.

On the basis of Theorem 1 on completeness of exponential systems it is possible to give a new proof of the Beurling–Malliavin Theorem on the radius of completeness. But it is impossible to name this new proof neither short, nor simple.

4 New results

Here we give only two results.

Theorem 3. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ and $\Gamma = \{\gamma_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ are two point sequences on $\mathbb{C}$ without limit points in $\mathbb{C}$. Suppose that there is $h \in \mathbb{R}$ such that $\operatorname{Im}\lambda_k \neq -h$ and $\operatorname{Im}\gamma_k \neq -h$ for all $k \in \mathbb{N}$, and*

$$\limsup_{t\to\pm\infty} \sum_{k\in\mathbb{N}} \left(\frac{|\operatorname{Im}\lambda_k + h|}{(t-\operatorname{Re}\lambda_k)^2 + (\operatorname{Im}\lambda_k + h)^2} - \frac{|\operatorname{Im}\gamma_k + h|}{(t-\operatorname{Re}\gamma_k)^2 + (\operatorname{Im}\gamma_k + h)^2} \right) < +\infty.$$

If the sequence Γ is a zero sequence for B_σ^∞ , then the sequence Λ is also a zero sequence for B_σ^∞.

If the system $\operatorname{Exp}^{i\Lambda}$ is complete in one of spaces $C(I_d)$ or $L^p(I_d)$, $p \geq 1$, then for any $\gamma', \gamma'' \notin \Gamma$, $\gamma' \neq \gamma''$ the system $\operatorname{Exp}^{i\Gamma\cup\{i\gamma'\}}$ is complete in $C(I_d)$ and in $L^p(I_d)$ for $p \geq 2$, and the system $\operatorname{Exp}^{i\Gamma\cup\{i\gamma', i\gamma''\}}$ is complete in $L^p(I_d)$ for $1 \leq p < 2$.

The notation of excesses for exponential system $\operatorname{Exp}^{i\Lambda}$ see in [10]. We denote its as $\operatorname{exc} i\Lambda$.

Corollary. *Let $\Lambda \cap \mathbb{R} = \varnothing$ and $\Gamma \cap \mathbb{R} = \varnothing$. If*

$$\limsup_{t\to\pm\infty} \sum_{k\in\mathbb{N}} \left| \operatorname{Im} \frac{\lambda_k - \gamma_k}{(t-\lambda_k)(t-\gamma_k)} \right| < +\infty,$$

then Λ and Γ can be zero sequences for B_σ^∞ only simultaneously.

Besides, $|\operatorname{exc} i\Lambda - \operatorname{exc} i\Gamma| \leq 1$ for $C(I_d)$ and $L^p(I_d)$ with $p \geq 2$, and $|\operatorname{exc} i\Lambda - \operatorname{exc} i\Gamma| \leq 2$ for $L^p(I_d)$ with $1 \leq p < 2$.

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